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Strong edge-coloring for jellyfish graphs

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ABSTRACT

A strong edge-coloring of a graph is a function that assigns to each edge a color such that two edges within distance two apart receive different colors. The *strong chromatic index* of a graph is the minimum number of colors used in a strong edge-coloring. This paper determines strong chromatic indices of cacti, which are graphs whose blocks are cycles or complete graphs of two vertices. The proof is by means of jellyfish graphs.

1. Introduction

The coloring problem considered in this article has restrictions on edges within distance two apart. The *distance* between two edges e and e' in a graph is the minimum k for which there is a sequence $e_0, e_1, \ldots, e_k$ of distinct edges such that $e = e_0$, $e' = e_k$, and e_{i-1} shares an end vertex with e_i for $1 \le i \le k$. A *strong edge-coloring* of a graph is a function that assigns to each edge a color such that any two edges within distance two apart receive different colors. A *color class* of a strong edge-coloring is the set of all edges using the same color. A *strong k-edge-coloring* is a strong edge-coloring using at most k colors. An *induced matching* is an edge set in which two distinct edges are of distance at least two. Finding a strong k-edge-coloring is equivalent to partitioning the edge set of the graph into k induced matchings. The *strong chromatic index* of a graph G, denoted by $\chi_s'(G)$, is the minimum k such that G admits a strong k-edge-coloring.

Strong edge-coloring was first studied by Fouquet and Jolivet [11,12] for cubic planar graphs. By a greedy algorithm, it is easy to see that $\chi_s'(G) \le 2\Delta^2 - 2\Delta + 1$ for any graph G of maximum degree Δ. Fouquet and Jolivet [11] established a Brooks type upper bound $\chi_s'(G) \le 2\Delta^2 - 2\Delta$, which is not true only for $G = C_5$ as pointed out by Shiu and Tam [26]. The following conjecture was posed by Erdős and Nešetřil [8,9] and revised by Faudree, Gyárfás, Schelp and Tuza [10]:

Conjecture 1. *If G is a graph of maximum degree Δ, then $\chi_s'(G) \le \Delta^2 + \lfloor \frac{\Delta}{2} \rfloor^2$.*

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For graphs with maximum degree $\Delta = 3$, Conjecture 1 was verified by Andersen [1] and by Horák, Qing and Trotter [15] independently. For $\Delta = 4$, while Conjecture 1 says that $\chi'_s(G) \le 20$, Horák [14] obtained $\chi'_s(G) \le 23$ and Cranston [7] proved $\chi'_s(G) \le 22$. Molloy and Reed [22] proved that for large Δ every graph of maximum degree Δ has $\chi'_s(G) \le 1.998\Delta^2$ using a probabilistic method. Mahdian [19] proved that for a C_4-free graph G, $\chi'_s(G) \le (2 + o(1))\Delta^2/\ln\Delta$. Faudree, Gyárfás, Schelp and Tuza [10] proved that for graphs where all cycle lengths are multiples of four, $\chi'_s(G) \le \Delta^2$. They mentioned that this result could probably be improved to a linear function of the maximum degree. Brualdi and Massey [2] improved the upper bound to $\chi'_s(G) \le \alpha\beta$ for such graphs, where α and β are the maximum degrees of the respective partitions. Nakprasit [23] proved that if G is bipartite and the maximum degree of one partite set is at most 2, then $\chi'_s(G) \le 2\Delta$. Chang and Narayanan [6] proved that $\chi'_s(G) \le 8\Delta - 6$ for chordless graphs G. This settles the above question by Faudree, Gyárfás, Schelp and Tuza [10] in the positive, since graphs with cycle lengths divisible by 4 are chordless graphs. They also established that $\chi'_s(G) \le 10\Delta - 10$ for 2-degenerate graphs G.

Strong edge-coloring on planar graphs is also extensively studied in the literature. Faudree, Gyárfás, Schelp and Tuza [10] asked whether $\chi'_s(G) \le 9$ if G is cubic planar. If this upper bound is proved to be true, it would be the best possible. Faudree, Gyárfás, Schelp and Tuza [10] used the Four-color Theorem to show that $\chi'_s(G) \le 4\Delta(G) + 4$ for any planar graph G of maximum degree Δ. They also exhibited a planar graph G whose strong chromatic index is $4\Delta(G) - 4$. Their proof also gives a consequence that $\chi'_s(G) \le 3\Delta$ for planar graphs G of girth at least 7. Chang, Montassier, Pecher and Raspaud [5] further proved that $\chi'_s(G) \le 2\Delta - 1$ for planar graphs G with large girth. Strong chromatic index for Halin graphs was first considered by Shiu, Lam and Tam [25] and then studied in [4,16,18,26]. For trees G they obtained that $\chi'_s(G) = \sigma(G)$, where

$$\sigma(G) := \max_{uv \in E(G)} \{d_G(u) + d_G(v) - 1\} \tag{1}$$

is an easy lower bound of $\chi'_s(G)$, that is,

$$\sigma(G) \le \chi'_s(G) \text{ for any graph } G. \tag{2}$$

An edge xy in a graph G is *σ-tight* if $d_G(x) + d_G(y) - 1 = \sigma(G)$. Liao [17] studied cacti, which are graphs whose blocks are cycles or complete graphs of two vertices. Notice that cacti are planar graphs and include trees. He established that for a cactus G, $\chi'_s(G) = \sigma(G)$ if the length of any cycle is a multiple of 6, $\chi'_s(G) \le \sigma(G) + 1$ if the length of any cycle is even, and $\chi'_s(G) \le \lfloor \frac{3\sigma(G)+1}{2} \rfloor$ in general. For other results on strong edge-coloring, see [3,13,20,21,24,27].

The purpose of this paper is to determine strong chromatic indices of cacti. The method is by means of jellyfish graphs to be introduced later. We first establish a decomposition theorem saying that the strong chromatic index of a graph is the maximum strong chromatic index of a block-jellyfish, which is a block together with edges with one vertex in the block and the other outside. Then we determine the strong chromatic index of a C_n-jellyfish, which is a graph obtained from the cycle C_n by attaching pendent edges to the cycle vertices.

2. Preliminary

For an integer $n \ge 3$, the *n-cycle* is the graph C_n with vertex set $V(C_n) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(C_n) = \{v_i v_{i+1} : 1 \le i \le n\}$, where $v_{n+1} = v_1$. More generally, when the indices of the vertices of an n-cycle are arithmetic expressions, they are considered to be taken modulo n.

A *cut-vertex* of a graph is a vertex whose removing results in a graph with more components than the old graph. A *block* of a graph is a maximal connected subgraph without cut-vertices in itself. Any two blocks of a graph have at most one vertex in common, and if they meet at one vertex, then it is a cut-vertex. For a block H of a graph G, any vertex $u \in V(G) - V(H)$ is adjacent to at most one vertex $v \in V(H)$, and if the vertex v exists then it is a cut-vertex of G. An *end block* is a block with exactly one cut-vertex. A *block graph* is a graph whose blocks are complete graphs. A *cactus* is a graph whose blocks are cycles or complete graphs of two vertices.

For a graph H, the *H-jellyfish* $H(p_v : v \in V(H))$ is the graph obtained from H by adding p_v new vertices adjacent to v for each vertex v in H. An edge joining a new vertex to v is called a *pendent edge* at v. A *block-jellyfish* of a graph G is the H-jellyfish H' for some block H of G, where the new vertices of H' are all vertices of $V(G) - V(H)$ having exactly one neighbor in $V(H)$. A block-jellyfish is *trivial* if it is an H-jellyfish for an end block H which is K_2, otherwise it is *non-trivial*.

Lemma 2. *If H is a subgraph of G, then $\chi'_s(H) \le \chi'_s(G)$.*

As any three consecutive edges in C_n use different colors in a strong edge-coloring, the following lemma is an easy consequence of parity checking.

Proposition 3. *If $n \ge 3$, then $\chi'_s(C_n) = 5$ for $n = 5$, $\chi'_s(C_n) = 3$ for n is a multiple of 3 and $\chi'_s(C_n) = 4$ otherwise.*

Notice that a trivial block-jellyfish H'_1 is a star; and if it is not a component, then it is a subgraph of a non-trivial block-jellyfish H'_2. By Lemma 2, $\chi'_s(H'_1) \le \chi'_s(H'_2)$.

Theorem 4. *Suppose G is a connected graph that is not a star. If G has exactly r non-trivial block-jellyfishes $G_1, G_2, \ldots, G_r$, then $\chi'_s(G) = \max_{1 \le i \le r} \chi'_s(G_i)$.*

Proof. Since the graphs G_i are subgraphs of G, by Lemma 2, $\chi_s'(G) \geq \max_{1\leq i\leq r} \chi_s'(G_i)$. Next, we shall prove by induction on r that $\chi_s'(G) \leq \max_{1\leq i\leq r} \chi_s'(G_i)$. In the case where $r = 1$, $G = G_1$ and so the inequality is clear. Assume $r \geq 2$. Suppose the corresponding block of G_i in G is H_i. Then there is some H_i, say H_1, which meets exactly one H_j at a cut-vertex of G. Let G' be obtained from G by deleting G_1 but keeping those vertices and edges in G_j undeleted. Then the non-trivial block-jellyfishes of G' are exactly $G_2, G_3, \ldots, G_r$. By the induction hypothesis, $\chi_s'(G') \leq \max_{2\leq i\leq r} \chi_s'(G_i)$. Color G' with $\chi_s'(G')$ colors. Since every two edges in $E(G_1) \cap E(G_j)$ are adjacent, meeting at the cut-vertex, we may assume that edges in $E(G_1) \cap E(G_j)$ are colored by $\{1, 2, \ldots, |E(G_1) \cap E(G_j)|\}$. On the other hand, since $d(e, e') > 2$ for any $e \in E(G_1) - E(G_j)$ and $e' \in E(G) - E(G_1)$, we can color edges in $E(G_1)$ by $\{1, 2, \ldots, \chi_s'(G_1)\}$. Hence, we have colored G by $\max\{\chi_s'(G'), \chi_s'(G_1)\} \leq \max_{1\leq i\leq r} \chi_s'(G_i)$ colors. □

As an easy consequence, we have the following results for block graphs.

Corollary 5. *If G is a block graph, then $\chi_s'(G) = \max\{|E(H)|: H$ is a non-trivial block-jellyfish of $G\}$.*

Proof. This follows from Theorem 4 and the fact that any two edges in H are of distance within two. □

Corollary 6. *If G is a C_3-jellyfish, then $\chi_s'(G) = |E(G)|$.*

Proof. This follows from Corollary 5 and the fact that G is the only non-trivial block-jellyfish of itself. □

Lemma 7. *If $G = H(p_v : v \in V(H))$ is an H-jellyfish such that $\{v : p_v \neq 0\} \subseteq X \cup Y$ for two independent sets X and Y, then $\chi_s'(G) \leq \chi_s'(H) + \max\{p_u + p_v : u \in X, v \in Y, uv \in E(H)\}$.*

Proof. Let $s = \max\{p_u + p_v : u \in X, v \in Y, uv \in E(H)\}$. For each vertex $u \in X$, color the pendent edges incident to u by $\{1, 2, \ldots, p_u\}$, and for each vertex $v \in Y$, color the pendent edges incident to v by $\{s - p_v + 1, s - p_v + 2, \ldots, s\}$. We verify that the coloring is legal. In fact, if a pendent edge uu' is within distance two from a pendent edge vv', then $uv \in E(H)$. The assumption $p_u + p_v \leq s$ gives $p_u < s - p_v + 1$, so uu' and vv' are colored differently. We then use $s + 1, s + 2, \ldots, s + \chi_s'(H)$ to color the edges of H. These give a strong edge-coloring of G and the lemma follows. □

Corollary 8. *If G is a C_n-jellyfish with even n, then $\chi_s'(G) \leq \sigma(G) + \chi_s'(C_n) - 3$.*

Proof. Let $X = \{v_i : i$ is odd$\}$ and $Y = \{v_i : i$ is even$\}$. The corollary follows from Lemma 7 and the fact that $\max_{1\leq i\leq n}\{p_i + p_{i+1}\} = \sigma(G) - 3$. □

Corollary 9 ([17]). *If G is a C_n-jellyfish with even n, then $\chi_s'(G) \leq \sigma(G) + 1$.*

Proof. This follows from Corollary 8 and the fact that $\chi_s'(C_n) \leq 4$. □

Corollary 10. *If G is a C_4-jellyfish, then $\chi_s'(G) = \sigma(G) + 1$.*

Proof. By Corollary 9, $\chi_s'(G) \leq \sigma(G) + 1$. On the other hand, consider a cycle edge xy such that $d_G(x) + d_G(y) - 1 = \sigma(G)$. Since the cycle edge not incident to x or y is within distance 2 from the edges incident to x or y, we have $\chi_s'(G) \geq \sigma(G) + 1$, so $\chi_s'(G) = \sigma(G) + 1$. □

Corollary 11 ([17]). *If G is a C_n-jellyfish with n a multiple of 6, then $\chi_s'(G) = \sigma(G)$.*

Proof. This follows from Corollary 8 and the fact that $\chi_s'(C_n) = 3$. □

Corollary 12. *Suppose G is a C_n-jellyfish with $d_G(v_j) = 2$ for some j. If $G \neq C_5$, then $\chi_s'(G) \leq \sigma(G) + 1$. If n is a multiple of 3, then $\chi_s'(G) = \sigma(G)$.*

Proof. Without loss of generality, we may assume that $j = n$. Let $X = \{v_i : i \neq n$ and i is odd$\}$ and $Y = \{v_i : i \neq n$ and i is even$\}$.

Since $\max_{1\leq i\leq n-1}\{p_i + p_{i+1}\} = \sigma(G) - 3$, $\chi_s'(G) \leq \chi_s'(C_n) + \sigma(G) - 3$ by Lemma 7. So, when $n \neq 5$, $\chi_s'(C_n) \leq 4$ implies that $\chi_s'(G) \leq \sigma(G) + 1$. When n is a multiple of 3, $\chi_s'(C_n) = 3$ implies that $\chi_s'(G) \leq \sigma(G)$ and then $\chi_s'(G) = \sigma(G)$.

For the case of $n = 5$, consider the C_5-jellyfish $H = C_5(\min\{p_i, 1\} : 1 \leq i \leq 5)$. Notice that every cycle vertex of H has at most one pendent edge. Then $\chi_s'(H) \leq 5$, since we can color the edges of H with 5 colors by coloring the pendent edge at v_i (if any) with the same color as the cycle edge $v_{i+2}v_{i+3}$, where the indices are taken modulo 5. Let $p_i' = p_i - \min\{p_i, 1\}$ for $1 \leq i \leq 5$. Notice that $p_5 = p_5' = 0$ and $G \neq C_5$. So, there is at least one $p_i \neq 0$ and then $\max_{1\leq i\leq 4}(p_i' + p_{i+1}') \leq \sigma(G) - 4$. Then G is the H-jellyfish $H(p_i' : 1 \leq i \leq 5)$, where the un-presented $p_u = 0$ for all leaves u of H. By Lemma 7, $\chi_s'(G) \leq \max_{1\leq i\leq 4}(p_i' + p_{i+1}') + \chi_s'(H) \leq \sigma(G) - 4 + 5 = \sigma(G) + 1$. □

3. Strong edge-coloring on cacti

The purpose of this section is to give the strong chromatic indices of cacti. Notice that a block-jellyfish of a cactus is either a K_2-jellyfish or a C_n-jellyfish. The strong chromatic index of a K_2-jellyfish is equal to its number of edges. So we only need to consider the case of C_n-jellyfish. Now suppose that G is a C_n-jellyfish. Notice that $G = C_n(p_1, p_2, \ldots, p_n)$, where $p_i = d_G(v_i) - 2$ for $1 \le i \le n$. A *rotation* of a C_n-jellyfish $G = C_n(p_1, p_2, \ldots, p_n)$ is a C_n-jellyfish $G' = C_n(p'_1, p'_2, \ldots, p'_n)$ with all $p'_i = p_{i+r}$ for a constant r, where the index in p_{i+r} is taken modulo n. As we have the values for cycles in Proposition 3, we may only consider the case of $\sigma(G) \ge 4$.

Theorem 13. *If G is a C_n-jellyfish of m edges with $\sigma(G) \ge 4$, then* $\chi'_s(G) =$

$$\begin{cases} m, & \text{if } n = 3;\\ \sigma(G)+1, & \text{if } n = 4;\\ \left\lceil \dfrac{m}{\lfloor n/2 \rfloor} \right\rceil, & \text{otherwise, if } n \text{ is odd with all } d_G(v_i) = d \text{ but } (n,d) \ne (7,3),\\ & \quad \text{or } \left\lceil \dfrac{m}{\lfloor n/2 \rfloor} \right\rceil \ge \sigma(G)+1;\\ \sigma(G)+1, & \text{otherwise, if } (n,d) = (7,3) \text{ with all } d_G(v_i) = d,\\ & \quad \text{or } n \not\equiv 0 \pmod 3 \text{ such that up to rotation}\\ & \qquad d_G(v_i) = \sigma(G) - 1 \text{ for } i \equiv 1 \pmod 3 \text{ with } 1 \le i \le 3\left\lfloor \dfrac{n}{3} \right\rfloor - 2,\\ & \quad \text{or } (n, \sigma(G)) = (10,4) \text{ with } d_G(v_i) = 3 \text{ for all odd or all even } i;\\ \sigma(G), & \text{otherwise.} \end{cases}$$

To prove the main theorem, we first establish a sequence of lemmas as follows.

Lemma 14. *If G is a C_n-jellyfish graph of m edges, then any color class of a strong edge-coloring has at most $\lfloor n/2 \rfloor$ edges and $\lceil \frac{m}{\lfloor n/2 \rfloor} \rceil \le \chi'_s(G)$.*

Proof. We claim that there are at most $\lfloor n/2 \rfloor$ edges using the same color in a strong edge-coloring of G. For $1 \le i \le n$, consider the set E_i consisting all edges incident to v_i or v_{i+1} except the edge $v_{i-1}v_i$. Then for a fixed color c, each E_i contains at most one edge colored by c. As each edge of G appears in exactly two sets in $E_1, E_2, \ldots, E_n$, there are at most $\lfloor n/2 \rfloor$ edges using the color c. Hence $\frac{m}{\lfloor n/2 \rfloor} \le \chi'_s(G)$ and so $\lceil \frac{m}{\lfloor n/2 \rfloor} \rceil \le \chi'_s(G)$. □

Lemma 15. *If n is even or $d_G(v_j) = 2$ for some j, then $\lceil \frac{m}{\lfloor n/2 \rfloor} \rceil \le \sigma(G)$. If n is odd and $d_G(v_i) = d$ for $1 \le i \le n$, then $\lceil \frac{m}{\lfloor n/2 \rfloor} \rceil = \sigma(G)$ for $2 \le d \le (n+1)/2$, $\lceil \frac{m}{\lfloor n/2 \rfloor} \rceil = \sigma(G) + 1$ for $(n+3)/2 \le d \le n$ and $\lceil \frac{m}{\lfloor n/2 \rfloor} \rceil \ge \sigma(G) + 2$ for $d \ge n+1$.*

Proof. If n is even or $d_G(v_j) = 2$ for some j, say $j = n$, then $m - 1 \le \sum_{i=1}^{\lfloor n/2 \rfloor} (d_G(v_{2i-1}) + d_G(v_{2i}) - 2) \le \lfloor n/2 \rfloor (\sigma(G) - 1)$, so $\lceil \frac{m}{\lfloor n/2 \rfloor} \rceil \le \sigma(G)$. If n is odd and $d_G(v_i) = d$ for $1 \le i \le n$, then $\lceil \frac{m}{\lfloor n/2 \rfloor} \rceil = 2d - 2 + \lceil \frac{2d-2}{n-1} \rceil = \sigma(G) - 1 + \lceil \frac{2d-2}{n-1} \rceil$, which is $\sigma(G)$ for $2 \le d \le (n+1)/2$, is $\sigma(G) + 1$ for $(n+3)/2 \le d \le n$ and is at least $\sigma(G) + 2$ for $d \ge n+1$. □

Lemma 16. *If G is a C_n-jellyfish with $d_G(v_i) = d \ge 3$ for $1 \le i \le n$, then*

$$\chi'_s(G) = \begin{cases} \sigma(G)+1, & \text{if } n = 4 \text{ or } (n,d) = (7,3);\\ \sigma(G), & \text{if } n \ge 6 \text{ is even};\\ \left\lceil \dfrac{m}{\lfloor n/2 \rfloor} \right\rceil, & \text{if } n \ge 3 \text{ is odd but } (n,d) \ne (7,3). \end{cases}$$

Proof. Case 1. $n = 4$. In this case, the lemma follows from Corollary 10.

Case 2. $(n, d) = (7, 3)$. In this case, an induced matching has at most 3 edges by Lemma 14. If $\chi'_s(G) \le 5$, then there are at least two color classes containing 2 cycle edges, each of which must be precisely of size 2. Then all color classes contain at most 13 edges, contradicting to the fact that G has 14 edges. Hence $\chi'_s(G) \ge 6$. Fig. 1 gives a strong 6-edge-coloring of G, so $\chi'_s(G) = 6 = \sigma(G) + 1$.

Case 3. $n \ge 6$ is even. In this case, by Corollary 11, we only need to consider the case of $n \not\equiv 0 \pmod 3$. For $1 \le i \le n$, let $e_i = v_iv_{i+1}$ and $f_{i,1}, f_{i,2}, \ldots, f_{i,d-2}$ be the pendent edges at v_i. The lemma follows from the fact that we may partition $E(G)$ into $\sigma(G) = 2d - 1$ induced matchings as follows:

$$\text{for } n \equiv 2 \pmod 3,\ \begin{cases} M_1 = \{f_{1,1}, e_3\} \cup \{e_i : 6 \le i \le n, i \equiv 0 \pmod 3\},\\ M_2 = \{f_{3,1}, e_5\} \cup \{e_i : 6 \le i \le n, i \equiv 2 \pmod 3\},\\ M_3 = \{f_{5,1}, e_2\} \cup \{e_i : 6 \le i \le n, i \equiv 1 \pmod 3\},\\ M_4 = \{e_1, e_4\} \cup \{f_{i,1} : 7 \le i \le n, i \equiv 1 \pmod 2\},\\ M_5 = \{f_{i,1} : 2 \le i \le n, i \equiv 0 \pmod 2\}; \end{cases}$$

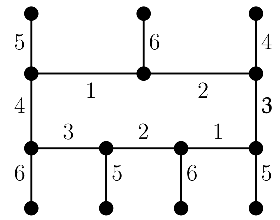

Fig. 1. A strong 6-edge-coloring of G.

for $n \equiv 1 \pmod 3$, $\begin{cases} M_1 = \{f_{1,1}, f_{6,1}, e_3\} \cup \{e_i : 6 \le i \le n, i \equiv 2 \pmod 3\}, \\ M_2 = \{f_{2,1}, f_{4,1}\} \cup \{e_i : 6 \le i \le n, i \equiv 0 \pmod 3\}, \\ M_3 = \{f_{3,1}, f_{5,1}\} \cup \{e_i : 6 \le i \le n, i \equiv 1 \pmod 3\}, \\ M_4 = \{e_1, e_4\} \cup \{f_{i,1} : 7 \le i \le n, i \equiv 1 \pmod 2\}, \\ M_5 = \{e_2, e_5\} \cup \{f_{i,1} : 7 \le i \le n, i \equiv 0 \pmod 2\}; \end{cases}$

for $2 \le j \le d-2$, $\begin{cases} M_{2j+2} = \{f_{i,j} : 1 \le i \le n, i \equiv 1 \pmod 2\}, \\ M_{2j+3} = \{f_{i,j} : 1 \le i \le n, i \equiv 0 \pmod 2\}. \end{cases}$

Case 4. $n \ge 3$ is odd but $(n, d) \ne (7, 3)$. In this case, $\lfloor n/2 \rfloor = (n-1)/2$, $m = n(d-1)$ and $\frac{m}{\lfloor n/2 \rfloor} = 2n(d-1)/(n-1)$. By Lemma 14, $\chi_s'(G) \ge \lceil \frac{m}{\lfloor n/2 \rfloor} \rceil$. We shall show the upper bound by considering two subcases.

Case 4-1. $3 \le d \le (n+1)/2$. In this subcase, we have $\lceil \frac{m}{\lfloor n/2 \rfloor} \rceil = 2d-1$. We will show that $\chi_s'(G) \le 2d-1$ and then $\chi_s'(G) = \lceil \frac{m}{\lfloor n/2 \rfloor} \rceil$. For any integer t and odd q with $1 \le q \le n/3$, let

$$I(t, q) = \{e_{t+3}, e_{t+6}, e_{t+9}, \ldots, e_{t+3q}\} \cup \{f_{t+3q+3}, f_{t+3q+5}, f_{t+3q+7}, \ldots, f_{t+n+1}\},$$

where the indices are taken modulo n and we denote every pendent edge at v_i by f_i for simplicity. This is an induced matching containing q cycle edges and $(n-3q)/2$ pendent edges. Since $5 \le 2d-1 \le n$, n is odd and $(n, d) \ne (7, 3)$, we can write n as a sum of $2d-1$ odd numbers $q_1, q_2, \ldots, q_{2d-1}$, each of which is no more than $n/3$. This can be done by choosing q_i's such that the gap between the maximum and the minimum is at most 2. Let $Q_0 = 0$ and $Q_i = \sum_{j=1}^{i} 3q_j$ for $1 \le i \le 2d-1$.

In the case where n is not a multiple of 3, we claim that the induced matchings $I(Q_{i-1}, q_i)$, for $1 \le i \le 2d-1$, partition $E(G)$. First, the cycle edges used are $e_3, e_6, e_9, \ldots, e_{3q_1}$; $e_{3q_1+3}, e_{3q_1+6}, e_{3q_1+9}, \ldots, e_{3q_1+3q_2}$; $\ldots$; $e_{Q_{2d-2}+3}, e_{Q_{2d-2}+6}, e_{Q_{2d-2}+9}, \ldots, e_{Q_{2d-2}+3q_{2d-1}} = e_{3n}$, which cover each cycle edge exactly once as n is not a multiple of 3. Secondly, the pendent edges used, viewing backward, are $f_{n+1}, f_{n-1}, f_{n-3}, \ldots, f_{3q_1+3}$; $f_{3q_1+1}, f_{3q_1-1}, f_{3q_1-3}, \ldots, f_{3q_1+3q_2+3}$; $\ldots$; $f_{Q_{2d-2}+1}, f_{Q_{2d-2}-1}, f_{Q_{2d-2}-3}, \ldots, f_{Q_{2d-2}+3q_{2d-1}+3} = f_{n+3}$, which cover pendent edges at each cycle vertex exactly $\frac{1}{n}\sum_{i=1}^{2d-1}(n-3q_i)/2 = d-2$ times.

In the case where n is a multiple of 3, we modify the above arguments as follows. We may assume that $n \ge 9$ as the case for $n = 3$ follows from Corollary 6. Since $5 \le 2d-1 \le n$, we may partition $2d-1$ into three odd numbers d_1, d_2, d_3 each satisfying $1 \le d_i \le n/3$. For $1 \le r \le 3$, partition $n/3$ into d_r odd numbers $q_{r,1}, q_{r,2}, \ldots, q_{r,d_r}$. We adopt similar arguments as above for the three parts separately, but consider rather $Q_{r,i} = r + \sum_{j=1}^{i} 3q_{r,j}$ for the rth part, $1 \le r \le 3$. The induced matchings in part r cover all the $n/3$ cycle edges e_i with $i \equiv r \pmod 3$, and the pendant edges $f_{r+n+1}, f_{r+n-1}, f_{r+n-3}, \ldots, f_{r+3(3n)+3}$, with a total number of a multiple of n. Similarly, pendant edges at each cycle vertex are also covered $d-2$ times.

Case 4-2. $d > (n+1)/2$. In this case, we partition the edges of G into two parts: the first part consists of the cycle edges together with $(n-3)/2$ pendent edges at each cycle vertex, and the second part consists of $d-(n+1)/2$ pendent edges at each cycle vertex. The first part has $m_1 = n(n-1)/2$ edges. By Case 4-1, it can be partitioned into n induced matchings. Next, we order the pendant edges in the second part as $h_1, h_2, \ldots, h_{m-m_1}$, where h_j is a pendant edge at cycle vertex v_i with $i \equiv 2j-1 \pmod n$. Notice that for any integer t and any integer $r \le (n-1)/2$, the set $\{h_{t+1}, h_{t+2}, \ldots, h_{t+r}\}$ is an induced matching. Hence the second part can be partitioned into $\lceil \frac{m-m_1}{(n-1)/2} \rceil$ induced matchings. Totally, the edges of G can be partitioned into $\lceil \frac{m}{(n-1)/2} \rceil$ as desired. □

We now consider the case where $d_G(v_j) = 2$ for some v_j, say v_n. By Corollary 12, $\chi_s'(G) = \sigma(G)$ or $\sigma(G) + 1$.

Lemma 17. *If $n \not\equiv 0 \pmod 3$ and G is a C_n-jellyfish such that $d_G(v_i) = \sigma(G) - 1$ for $i \equiv 1 \pmod 3$ and $1 \le i \le 3\lfloor n/3 \rfloor - 2$, then $\chi_s'(G) = \sigma(G) + 1$.*

Proof. First, the assumption gives that $d_G(v_j) = 2$ for $j = n$ or $j \not\equiv 1 \pmod 3$ with $1 \le j \le 3\lfloor n/3 \rfloor - 1$. By Corollary 12, $\chi_s'(G) \le \sigma(G) + 1$. Suppose to the contrary that G had a strong edge-coloring using $\sigma(G)$ colors. Then for each $i \equiv 1 \pmod 3$ with $1 \le i \le 3\lfloor n/3 \rfloor - 2$, the $\sigma(G) - 3$ pendent edges at v_i, e_{i-1}, e_i, together with e_{i-2} (respectively, e_{i+1}) would use all the $\sigma(G)$ colors. It follows that e_{i-2} and e_{i+1} would use the same color. Hence $e_{n-1}, e_2, e_5, \ldots, e_{3\lfloor n/3 \rfloor - 1}$ would all use the same color. Since $n \not\equiv 0 \pmod 3$, e_{n-1} and $e_{3\lfloor n/3 \rfloor - 1}$ are two distinct edges within distance two, which leads to a contradiction. □

Lemma 18. *If G is a C_{10}-jellyfish such that $d_G(v_i) = \sigma(G) - 1 = 3$ for all odd i, then $\chi_s'(G) = \sigma(G) + 1 = 5$.*

Proof. First, the assumption gives that $d_G(v_j) = 2$ for all even j. By Corollary 12, $\chi_s'(G) \le \sigma(G) + 1$. Suppose to the contrary that G had a strong edge-coloring using $\sigma(G) = 4$ colors. Then for each odd i, the $\sigma(G) - 3 = 1$ pendent edge at v_i, e_{i-1}, e_i,

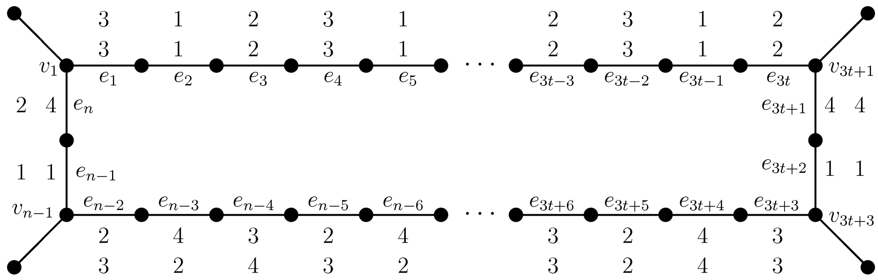

Fig. 2. Inner labels are for $n \equiv 1 \pmod 3$ and outer labels are for $n \equiv 2 \pmod 3$.

together with e_{i-2} (respectively, e_{i+1}) would use all the $\sigma(G)$ colors. This gives that e_{i-2} and e_{i+1} use the same color. Since we only had 4 colors for the 10 cycle edges, there would be one color used for at least 3 edges. But a color should appear in a pair of edges as shown above. This color would then be used for at least 4 cycle edges, which is impossible. □

Lemma 19. *If G is a C_n-jellyfish with $\sigma(G) = 4$, then $\chi'_s(G) = \sigma(G)$ except that $\chi'_s(G) = \sigma(G) + 1$ when, up to rotation, $n \not\equiv 0 \pmod 3$ such that $d_G(v_i) = 3$ for $i \equiv 1 \pmod 3$ with $1 \le i \le 3\lfloor \frac{n}{3} \rfloor - 2$ or $n = 10$ such that $d_G(v_i) = 3$ for all odd i.*

Proof. If n is a multiple of 3, then the lemma follows from Corollary 12. Now assume that $n \not\equiv 0 \pmod 3$. The exceptional cases follow from Lemmas 17 and 18.

Up to rotation, we may assume that $1 = i_1 < i_2 < \cdots < i_s$ are all the indices for which v_{i_r} is of degree 3. For $1 \le r \le s$, the path P_r from v_{i_r} to $v_{i_{r+1}}$ consists of $n_r = i_{r+1} - i_r$ cycle edges, where $i_{s+1} = n + 1$. Using this notion, the C_n-jellyfish G is completely determined by the sequence $n_1, n_2, \ldots, n_s$. Notice that the first exceptional case is the same as that all $n_r = 3$ except exactly one $n_r \in \{2, 4, 5\}$ or exactly two consecutive $n_r = 2$, and the second exceptional case is the same as that $n = 10$ and all $n_r = 2$. We consider cases other than the two exceptional cases. Since the cases for $n = 4, 5$ are included in the first exceptional case, and 6 is a multiple of 3, we may assume $n \ge 7$. The aim is to find a strong 4-edge-coloring for G. By adding suitable pendent edges, we may assume that all $n_r \in \{2, 3\}$ and there are two non-consecutive $n_r = 2$.

If there is at least one $n_r = 3$, then up to rotation we may assume that $n_s = 2$, and there exists some $t \le s - 1$ such that $n_r = 3$ for all $1 \le r \le t$ and $n_{t+1} = 2$. Otherwise, if all $n_r = 2$, then $s \ge 4$, $s \ne 5$, and we choose $t = 0$. We define an edge-coloring on cycle edges first as in Fig. 2.

These colors for the cycle edges satisfy the following two conditions.

(i) Any two distinct cycle edges within distance two receive distinct colors.
(ii) The two cycle edges with distance exactly two from a pendent edge receive a same color.

By (ii), the four cycle edges within distance two from a pendent edges use only 3 colors. Hence we may color any pendent edge with the remaining color to form a strong 4-edge-coloring of G. □

Lemma 20. *If G is a C_n-jellyfish with $\sigma(G) \ge 4$ and $d_G(v_j) = 2$ for some j, then $\chi'_s(G) = \sigma(G)$ except that $\chi'_s(G) = \sigma(G) + 1$ when, up to rotation, $n \not\equiv 0 \pmod 3$ such that $d_G(v_i) = \sigma(G) - 1$ for $i \equiv 1 \pmod 3$ with $1 \le i \le 3\lfloor \frac{n}{3} \rfloor - 2$ or $n = 10$ such that $d_G(v_i) = \sigma(G) - 1 = 3$ for all odd i.*

Proof. The exceptional cases follow from Lemmas 17 and 18. We shall prove the lemma by induction on $\sigma(G)$. The case of $\sigma(G) = 4$ follows from Lemma 19. Now assume that $\sigma(G) \ge 5$.

A *run* is a maximal sequence $v_i, v_{i+1}, \ldots, v_{i+j}$ of cycle vertices in which every vertex is of degree at least 3. The *even-half* (respectively, *odd-half*) of the run is the vertices v_{i+r} with $0 \le r \le j$ and r even (respectively, odd). Notice that an even-half of a run is always non-empty, while an odd-half is empty if and only if $j = 0$. Consider a C_n-jellyfish G' obtained from G by deleting a pendent edge at each vertex of exactly one of the even-half or the odd-half of each run. Then $\sigma(G) = \sigma(G') + 1$ and $\chi'_s(G) \le \chi'_s(G') + 1$ as the deleted edges form an induced matching.

Suppose that G' is not in the exceptional cases. By the induction hypothesis, $\chi'_s(G') = \sigma(G')$. Then $\chi'_s(G) \le \chi'_s(G') + 1 = \sigma(G') + 1 = \sigma(G)$, so $\chi'_s(G) = \sigma(G)$. Now we may assume that G' is in the exceptional cases. If there is a run of length one in G' obtained from some run of length not one in G, then we change to delete the other half of this run in G and obtain a new G' which is not in the exceptional cases. Now every run of length one in G' is obtained from a run of length one in G, and since G' is in the exceptional cases but not G, it must be that $n = 10$ and $d_G(v_i) = \sigma(G) - 1 = 4$ for all odd i. Then $\chi'_s(G) = \sigma(G) = 5$ as shown in Fig. 3. □

Having the above lemmas established, we are now ready to prove Theorem 13. For the case of $n = 3$, the theorem follows from Corollary 6. For the case of $n = 4$, the theorem follows from Corollary 10. Now we may assume that $n \ge 5$.

If $d_G(v_j) = 2$ for some j, then $\lceil \frac{m}{\lfloor n/2 \rfloor} \rceil \le \sigma(G)$ by Lemma 15, so the third case of the theorem does not happen. The theorem then follows from Lemma 20.

We now consider the case where $d_G(v_i) \ge 3$ for all i. There are two subcases to be considered depending on the parity of n.

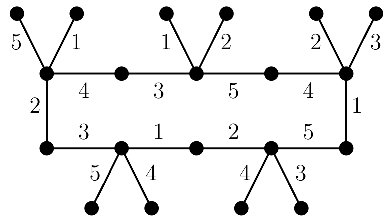

Fig. 3. The C_{10}-jellyfish G with $d_G(v_i) = \sigma(G) - 1 = 4$ for all odd i.

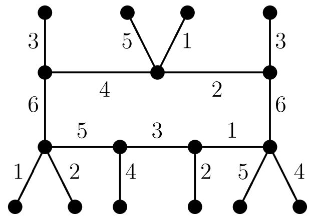

Fig. 4. The graph G_{s-1} for $(n, d_s) = (7, 3)$.

We first consider the subcase where n is even. By Lemma 15, $\lceil \frac{m}{\lfloor n/2 \rfloor} \rceil \le \sigma(G)$, so the third case of the theorem does not happen. We then only need to prove that $\chi_s'(G) = \sigma(G)$. Let H be the C_n-jellyfish with each cycle vertex v_i of degree 3. By Lemma 16, $\chi_s'(H) = 5$. Then G is obtained from H by adding $p_{v_i} = d_G(v_i) - 3$ pendent edges at v_i for $1 \le i \le n$. Let $X = \{v_i : 1 \le i \le n \text{ and } i \text{ odd}\}$ and $Y = \{v_i : 1 \le i \le n \text{ and } i \text{ even}\}$. Then $\max\{p_u + p_v : u \in X, v \in Y, uv \in E(H)\} = \sigma(G) - 5$. By Lemma 7, $\chi_s'(G) \le \chi_s'(H) + \max\{p_u + p_v : u \in X, v \in Y, uv \in E(H)\} \le 5 + \sigma(G) - 5 = \sigma(G)$ and so $\chi_s'(G) = \sigma(G)$.

Next we consider the second subcase when n is odd. If all cycle edges $v_i v_{i+1}$ are tight, then $d_G(v_i) + d_G(v_{i+1}) - 1 = d_G(v_{i+1}) + d_G(v_{i+2}) - 1$ and so $d_G(v_i) = d_G(v_{i+2})$ for all i. Since n is odd, all $d_G(v_i)$ are equal.

Suppose, up to rotation, that $v_n v_1$ is a non-tight edge. Consider the C_n-jellyfish graph G_1 obtained from $G_0 := G$ by deleting one pendent edge at v_i for all even i. Then $\sigma(G_1) = \sigma(G_0) - 1$ and G_1 has $m_1 = m - \lfloor n/2 \rfloor$ edges. Since we can use one color for the deleted edges, $\chi_s'(G_0) \le \chi_s'(G_1) + 1$. Repeating the same process gives that there is an integer $s \ge 0$ and C_n-jellyfish graphs $G_0, G_1, \ldots, G_s$ such that $\sigma(G_r) \ge 4$, $\sigma(G_r) = \sigma(G) - r$, G_r has $m_r = m - r\lfloor n/2 \rfloor$ edges, $\chi_s'(G) \le \chi_s'(G_r) + r$ for $0 \le r \le s$, and either $d_{G_s}(v_j) = 2$ for some j or else $d_{G_s}(v_i)$ is a constant d_s for all i.

For the former case, G_{s-1} has the property that all cycle vertices have degree at least 3. But after deleting $(n-1)/2$ pendent edges, the resulting graph G_s has some cycle vertex v_j with degree 2. It then must be the case that G_s is not in the exceptional cases in Lemma 20. Hence $\chi_s'(G_s) = \sigma(G_s)$ and $\chi_s'(G) \le \chi_s'(G_s) + s = \sigma(G_s) + s = \sigma(G)$. By Lemma 15, $\frac{m_s}{\lfloor n/2 \rfloor} \le \sigma(G_s)$, so $\frac{m}{\lfloor n/2 \rfloor} = \frac{m_s}{\lfloor n/2 \rfloor} + s \le \sigma(G_s) + s = \sigma(G)$. It follows that G fits the fifth case.

Now we may assume that $d_{G_s}(v_i)$ is a constant d_s for all i. If $(n, d_s) \ne (7, 3)$, then by Lemma 16, $\chi_s'(G_s) = \lceil \frac{m_s}{\lfloor n/2 \rfloor} \rceil$ and so $\chi_s'(G) \le \chi_s'(G_s) + s = \lceil \frac{m_s}{\lfloor n/2 \rfloor} \rceil + s = \lceil \frac{m}{\lfloor n/2 \rfloor} \rceil$. By Lemma 14, $\chi_s'(G) = \lceil \frac{m}{\lfloor n/2 \rfloor} \rceil$. If $(n, d_s) = (7, 3)$, then G_{s-1} must be the graph as in Fig. 4 from which we conclude that $\chi_s'(G_{s-1}) = \sigma(G_{s-1})$. Then $\chi_s'(G) \le \chi_s'(G_{s-1}) + s - 1 = \sigma(G_{s-1}) + s - 1 = \sigma(G)$. Notice that $\lceil \frac{m_s}{\lfloor n/2 \rfloor} \rceil = 5 = \sigma(G_s)$ and $\lceil \frac{m}{\lfloor n/2 \rfloor} \rceil = \lceil \frac{m_s}{\lfloor n/2 \rfloor} \rceil + s = \sigma(G_s) + s = \sigma(G)$, so G fits the fifth case.

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References

[1] L.D. Andersen, The strong chromatic index of a cubic graph is at most 10, Discrete Math. 108 (1992) 231–252.
[2] R.A. Brualdi, J.J.Q. Massey, Incidence and strong edge colorings of graphs, Discrete Math. 122 (1993) 51–58.
[3] K. Cameron, Induced matchings, Discrete Appl. Math. 24 (1989) 97–102.
[4] G.J. Chang, D.D.-F. Liu, Strong edge-coloring for cubic halin graphs, Discrete Math. 312 (2012) 1468–1475.
[5] G.J. Chang, M. Montassier, A. Pecher, A. Raspaud, Strong chromatic index of planar graphs with large girth, Discuss. Math. Graph Theory 34 (2014) 723–733.
[6] G.J. Chang, N. Narayanan, Strong chromatic index of 2-degenerate graphs, J. Graph Theory 73 (2013) 119–126.
[7] D. Cranston, Strong edge-coloring graphs with maximum degree 4 using 22 colors, Discrete Math. 306 (2006) 2772–2778.
[8] P. Erdős, Problems and results in combinatorial analysis and graph theory, Discrete Math. 72 (1988) 81–92.
[9] P. Erdős, J. Nešetřil, [Problem], in: G. Halász, V.T. Sós (Eds.), Irregularities of Partitions, Springer, Berlin, 1989, pp. 162–163.
[10] R.J. Faudree, A. Gyárfás, R.H. Schelp, Zs. Tuza, The strong chromatic index of graphs, Ars Combin. 29B (1990) 205–211.
[11] J.L. Fouquet, J. Jolivet, Strong edge-coloring of graphs and applications to multi-k-gons, Ars Combin. 16A (1983) 141–150.
[12] J.L. Fouquet, J. Jolivet, Strong edge-coloring of cubic planar graphs, in: Progress in Graph Theory (Waterloo 1982), 1984, pp. 247–264.

[13] M.C. Golumbic, M. Lewenstein, New results on induced matchings, Discrete Appl. Math. 101 (2000) 157–165.
[14] P. Horák, The strong chromatic index of graphs with maximun degree four, Contemp. Methods Graph Theory (1990) 399–403.
[15] P. Horák, H. Qing, W.T. Trotter, Induced matchings in cubic graphs, J. Graph Theory 17 (1993) 151–160.
[16] H.-H. Lai, K.-W. Lih, P.-Y. Tsai, The strong chromatic index of Halin graphs, Discrete Math. 312 (2012) 1536–1541.
[17] S.-T. Liao, The Strong Chromatic Index of Cacti (Master thesis), Department of Mathematics, National Taiwan University, Taipei, Taiwan, 2012, June.
[18] K.-W. Lih, D.D.-F. Liu, On the strong chromatic index of cubic Halin graphs, Appl. Math. Lett. 25 (2012) 898–901.
[19] M. Mahdian, The strong chromatic index of C_4-free graphs, Random Structures Algorithms 17 (2000) 357–375.
[20] M. Mahdian, On the computational complexity of strong edge coloring, Discrete Appl. Math. 118 (2002) 239–248.
[21] M. Maydanskiy, The incidence coloring conjecture for graphs of maximum degree 3, Discrete Math. 292 (2005) 131–141.
[22] M. Molloy, B. Reed, A bound on the strong chromatic index of a graph, J. Combin. Theory Ser. B 69 (1997) 103–109.
[23] K. Nakprasit, A note on the strong chromatic index of bipartite graphs, Discrete Math. 308 (2008) 3726–3728.
[24] M.R. Salavatipour, A polynomial time algorithm for strong edge coloring of partial k-trees, Discrete Appl. Math. 143 (2004) 285–291.
[25] W.C. Shiu, P.C.B. Lam, W.K. Tam, On strong chromatic index of Halin graphs, J. Combin. Math. Combin. Comput. 57 (2006) 211–222.
[26] W.C. Shiu, W.K. Tam, The strong chromatic index of complete cubic Halin graphs, Appl. Math. Lett. 22 (2009) 754–758.
[27] J. Wu, W. Lin, The strong chromatic index of a class of graphs, Discrete Math. 308 (2008) 6254–6262.